

Weak and Strong Extensions of Ordinary Differential Operators

ALBERTO BAIDER

*Department of Mathematics, Hunter College of the City University of New York,
New York, New York 10021**Submitted by J. P. LaSalle*

Let A be a second order differential operator with positive leading term defined on an interval J of R . In this paper we study conditions for the equality $D_0(A) = D_1(A)$ to hold. Here $D_0(A)$ and $D_1(A)$ are the domains of the minimal and maximal extensions of A respectively. Under the general assumption that $A(1)$ and $A^*(1)$ are bounded above it is proven that under certain conditions $D_0(A) = D_1(A)$ if functions which are constant near the boundaries of J are in $D_0(A) \cap D_0(A^*)$ whenever they are in $D_1(A) \cap D_1(A^*)$. In particular if A is formally selfadjoint and $1 \in D_1(A)$ then $D_0(A) = D_1(A)$ if and only if $1 \in D_0(A)$. When the measure of J is infinite at both ends $D_0(A)$ is always equal to $D_1(A)$. This fact is used to show that the leading term of A as well as its terminal coefficient can be chosen arbitrarily (although not independently of one another) in such a way that the equality $D_0 = D_1$ holds.

1. INTRODUCTION

Let A be a second order differential operator defined on an open interval $J: -\infty \leq l_0 < l_1 \leq \infty$. We assume that the leading term of A is strictly positive on J . In addition there is given a positive smooth (C^∞) density $\mu(x)$ on J , in terms of which we define the space $D_1 = D_1(A)$ by

$$D_1 = \{u \in L_\mu^2(J) \mid Au \in L_\mu^2(J)\}.$$

Here Au is understood in the sense of distributions. D_1 is a Hilbert space when given the topology of the graph norm of A :

$$\|u\|_A^2 = \|u\|^2 + \|Au\|^2.$$

In this paper we study conditions for the density of C_0^∞ in D_1 . This is referred to as the equality $D_0 = D_1$ where D_0 denotes the graph closure of C_0^∞ in D_1 . In the terminology of Dunford and Schwartz [1], this will be the case if and only if A has no boundary values at either end point of J . When A is selfadjoint this has been classically known as the limit point case of H. Weyl [6]. Although for technical reasons we treat both end points simultaneously, it should be noted that every result in this paper has a "one-sided" analog. We will refrain from

stating the corresponding facts explicitly. Nevertheless, they will be used in examples. If l is an end point of J we will use the phrase " $D_0 = D_1$ at l " without further comment.

Our main result is that for this equality to hold, it is necessary and sufficient (under conditions to be stated below) that functions which are constant near the boundaries of J should be in D_0 whenever they are in D_1 . Thus in particular if A is formally selfadjoint the μ -measure of J is finite and $A(1)$ is bounded above as well as in L_μ^2 then $D_0 = D_1$ if and only if $1 \in D_0$. This is surprising in view of the fact that D_1 will generally contain unbounded functions whenever $1 \in D_0$; for if $D_1 \subset L^\infty$ it follows from the closed graph theorem that the inclusion must be continuous; therefore if 1 were in D_0 it would be uniformly approximable by C_0^∞ functions which is clearly impossible. Thus our result implies the truth of the statement: if 1 is approximable by C_0^∞ (in the graph norm of A) then D_1 must contain unbounded functions which are in turn approximable by C_0^∞ .

Another consequence of our work is a generalization to the nonselfadjoint case of the classical result [6, 4] that if J has infinite measure at both ends then $D_0 = D_1$ is always true, provided that the terminal coefficients $A(1)$ and $A^*(1)$ of A and A^* respectively are bounded above.

The added generality obtained by considering arbitrary measures, although in a sense illusory, is important even when considering a problem originally stated in the more familiar terms of the Lebesgue measure for example. In fact if one can find a positive density $\mu(x)$ on J such that the μ -measure of J is infinite at both ends and such that $\mu^{-1/2}A(\mu^{1/2})$ as well as $\mu^{-1/2}A^*(\mu^{1/2})$ are bounded above then $D_0 = D_1$. Using this technique one can derive a whole class of explicit sufficient conditions for $D_0 = D_1$. We limit ourselves to obtain some necessary and some sufficient conditions relating mostly to the anti-symmetric part of A and to the condition $1 \in D_1(A) \cap D_1(A^*)$. Finally we use the same idea to prove the following interesting fact: let $A_{p,q}$ denote the operator

$$\frac{1}{\mu} \frac{d}{dx} p \frac{d}{dx} + q$$

where μ is a fixed density and q is strictly positive. The question arises: what kind of coefficients are permissible in the sense that $D_0(A_{p,q}) = D_1(A_{p,q})$? We prove that p can be arbitrary (there is always some q that will do) and that when the μ -measure of J is infinite at both ends the termination coefficient q can also be taken to be arbitrary.

2. THE MAIN THEOREM

We assume that A is a polynomial of degree 2 in d/dx with real C^∞ coefficients and a strictly positive leading part. There is no loss if we write A in the form

$$A = \mu^{-1}(d/dx) p(d/dx) + \mu^{-1}r(d/dx) + q, \quad (2.1)$$

with p, μ positive everywhere. Note that we do not include μ in the definition of q since this coefficient plays a special role in our considerations; in addition it can be defined intrinsically as $A(1)$. As we know, A has a formal adjoint A^* defined by the identity

$$(A\phi, \psi) = (\phi, A^*\psi), \quad \phi, \psi \in C_0^\infty(J)$$

where the inner product is taken with respect to the measure $\mu(x) dx$:

$$(\phi, \psi) = \int_J \phi \bar{\psi} \mu(x) dx.$$

At this point it should be mentioned that all the function spaces involved are real, although the results extend to their complexifications, provided the coefficients of A are kept real. We define $D_0(A^*)$, $D_1(A^*)$ by the same prescription given for A in the introduction and we remark that $D_1(A^*)$ is the domain of the (Hilbert space) adjoint of $A \upharpoonright D_0(A)$. Similarly $D_0(A^*)$ is the domain of the adjoint of $A \upharpoonright D_1(A)$. If l is an end point of J it is convenient to introduce a function $u_l \in C^\infty(J)$ which satisfies:

$$\begin{aligned} u_l &= 1 && \text{near } l \text{ (i.e., on an interval having } l \text{ as a limit point)} \\ &= 0 && \text{near the other end point of } J. \end{aligned}$$

The behaviour of u_l on compact subsets of J is immaterial to us.

PROPOSITION 2.1. *Let $u \in D_i(A)$, $v \in D_j(A^*)$, $0 \leq i + j \leq 1$. Then*

$$\lim_{x \rightarrow l} (p(u'v - uv') + ruv)(x) = 0. \quad (2.2)$$

Proof. The hypothesis implies that $(Au, v) = (u, A^*v)$. Now we may assume that $u = 0$ near $\partial J \setminus l$ since $u \in D_i(A)$ implies $uu_l \in D_i(A)$ as is readily seen from the identity

$$A(uu_l) = u_l Au + u(A - A(1))u_l + 2\mu^{-1}pu'u'_l$$

and the well-known fact that $D_1(A) \subset H_2^{\text{loc}}(J)$. Thus integration by parts yields

$$\begin{aligned} 0 &= (Au, v) - (u, A^*v) = \lim_{x \rightarrow l} \int_{\partial J \setminus l}^x (vAu - uA^*v) \mu dt \\ &= \lim_{x \rightarrow l} (p(u'v - uv') + ruv)(x) \end{aligned}$$

which is the desired formula.

An appropriate choice of u and/or v in (2.2) gives:

COROLLARY 2.2. (i) *If $u_l \in D_1(A^*)$ then*

$$\lim_{x \rightarrow l} (pu' + ru)(x) = 0, \quad u \in D_0(A). \quad (2.3)$$

(ii) If $u_l \in D_0(A^*)$ then

$$\lim_{x \rightarrow l} (pu' + ru)(x) = 0, \quad u \in D_1(A). \quad (2.4)$$

(iii) If $u_l \in (D_0(A) \cap D_1(A^*)) \cup [D_0(A^*) \cap D_1(A)]$ then

$$\lim_{x \rightarrow l} r(x) = 0. \quad (2.5)$$

For the remainder of this section it will be convenient to introduce the following terminology:

DEFINITION 2.3. An end point l of J such that $u_l \in D_1(A) \cap D_1(A^*)$ will be said to be *finite*.

It is clear that l is finite if and only if the measure of J is finite near l and in addition both $A(1)$ and $A^*(1)$ are in L_μ^2 near l . One easily checks that the latter are fulfilled if and only if q and $\mu^{-1}r'$ are in L_μ^2 near l (cf. 2.1). The following is a direct consequence of Corollary 2.2(iii):

COROLLARY 2.3. A necessary condition for the equality $D_0(A) = D_1(A)$ to hold is that $\lim_{x \rightarrow l} r(x) = 0$ whenever l is finite.

We now come to the main theorem of this section. We shall make the following assumptions.

$$A(1) \text{ and } A^*(1) \text{ are bounded above.} \quad (2.6)$$

$$\text{If } l \text{ is an end point of } J \text{ and } \left| \int_{x_0}^l \mu(y) dy \right| < \infty, \text{ then } l \text{ is finite.} \quad (2.7)$$

$$\text{If } l \text{ is finite and } r(x) \neq 0 \text{ between } x_0 \text{ and } l \text{ then} \quad (2.8)$$

$$\left| \int_{x_0}^l \left| r^{-1} \int_y^l \mu(x) dx \right|^2 \mu(y) dy \right| = \infty.$$

The divergence of the integral in (2.8) is obviously a condition on how fast r vanishes near an end point which is finite. For example if $r = x^\alpha$, $0 < x$ and $\mu = 1$ then r satisfies (2.8) if and only if $\alpha \geq \frac{3}{2}$. That some condition of this sort is needed follows from Corollary 2.3.

THEOREM 2.4. Let A satisfy (2.6), (2.7), and (2.8). Then $D_0 = D_1$ if and only if $u_l \in D_0(A) \cap D_0(A^*)$ whenever l is a finite end point of J .

For the proof we will need two lemmas. Let

$$\lambda_0 = \sup \frac{1}{2}(A(1) + A^*(1)). \quad (2.9)$$

LEMMA 2.5. *The following a priori estimates for A and A^* hold:*

$$\|\lambda u - Au\| \geq (\lambda - \lambda_0) \|u\|, \quad u \in D_0(A), \quad \lambda \in R, \quad (2.10)$$

$$\|v - A^*v\| \geq (\lambda - \lambda_0) \|v\|, \quad v \in D_0(A^*), \quad \lambda \in R. \quad (2.11)$$

Proof. Both sides of (2.10) and (2.11) are continuous in the graph norms of A and A^* respectively. Therefore it suffices to consider u, v in C_0^∞ . A computation shows that for ϕ, ψ smooth

$$A(\phi\psi) = \phi A\psi + \psi A\phi - \phi\psi A(1) + 2\mu^{-1}p\phi'\psi'.$$

In particular if $\phi = \psi$ we get:

$$\phi A\phi = 1/2(A(\phi^2) + \phi^2 A(1)) - \mu^{-1}p(\phi')^2$$

which in turn implies when $\phi \in C_0^\infty$

$$(\phi, A\phi) = (\phi A\phi, 1) = (\phi^2, 1/2(A(1) + A^*(1))) - (\mu^{-1}p(\phi')^2, 1).$$

Since $p > 0$ and $1/2(A(1) + A^*(1)) \leq \lambda_0$ we have

$$(\phi, A\phi) \leq \lambda_0 \|\phi\|^2.$$

An application of Schwartz's inequality gives the lemma.

LEMMA 2.6. *Let A be an arbitrary differential operator satisfying (2.10) and (2.11). Then the following are equivalent:*

- (i) $D_0(A) = D_1(A)$
- (ii) $D_0(A^*) = D_1(A^*)$
- (iii) For all $\lambda > \lambda_0$ $\text{Ker}(\lambda - A) \cap D_1(A) = \text{Ker}(\lambda - A^*) \cap D_1(A^*) = 0$
- (iv) For some $\lambda > \lambda_0$ $\text{Ker}(\lambda - A) \cap D_1(A) = \text{Ker}(\lambda - A^*) \cap D_1(A^*) = 0$.

Proof. (i) and (ii) are clearly equivalent and together they imply (iii). Since (iv) is a particular case of (iii) we prove that (iv) implies (i). Let $u \in D_1(A)$. With λ given by (iv) consider $f = \lambda u - Au$. We would like to solve this equation in D_0 , for if u_0 is such a solution then $u - u_0 \in D_1(A) \cap \text{Ker}(\lambda - A) = 0$ which proves that $u \in D_0(A)$. Now the operator A^* with domain $D_1(A^*)$ is the L_μ^2 adjoint of A with domain $D_0(A)$ and since $\text{Ker}(\lambda - A^*) \cap D_1(A^*) = 0$ it follows that the range of $\lambda - A$ on $D_0(A)$ is dense in L_μ^2 . Since this operator is closed and satisfies the estimate (2.10) it follows that its range is closed and therefore it is the whole of L_μ^2 which proves the lemma.

Proof of Theorem 2.4. The only thing to prove is the sufficiency. Pick λ_1 such that

$$\lambda_1 > \max\{\sup_j A(1), \sup_j A^*(1)\}.$$

Note that $\lambda_1 > \lambda_0$. We shall prove that if $\lambda_1 u - Au = 0$ then u is not in L_μ^2 . Since the same applies to A^* , this is all we need in view of Lemmas 2.5 and 2.6. Now suppose that $u \in L_\mu^2$ and that $u \neq 0$. We may assume by changing u to $-u$ if necessary that $u(x_0) > 0$ for some $x_0 \in J$; note that $u \in C^\infty$. We next remark that by our choice of λ_1 the operator $\lambda_1 - A$ satisfies the following strong form of the maximum principle: If $u(x_1)$ and $u'(x_1)$ are positive for some x_1 then u' is positive to the right of x_1 and consequently u is strictly increasing there. If $u'(x_1) < 0$ and $u(x_1) > 0$ then u' is negative to the left of x_1 and u is strictly decreasing there. Moreover if $u'(x_1) = 0$ then u' is positive to the right of x_1 and negative to its left. This means that by changing x_0 a little if necessary we may assume that $u'(x_0) \neq 0$. We will consider the case $u'(x_0) > 0$. The other is entirely similar. Consider the end point $l_1 > x_0$. Since $u \in L_\mu^2$ and $u(x) > u(x_0) > 0$ when $x > x_0$, it follows that the measure of the interval (x_0, l_1) must be finite. Now the assumption (2.7) implies that l_1 is a finite end point in the sense of Definition 2.3. Our hypothesis is that for such a point, $u_{l_1} \in D_0(A^*) \cap D_0(A)$. From Corollary 2.2(ii) we get that $(pu' + ru)(x) \rightarrow 0$ as $x \rightarrow l_1$. Thus integration from $y \geq x_0$ to l_1 gives:

$$\begin{aligned} -(pu' + ru)(y) &= \int_y^{l_1} (\lambda_1 - (q - r')) u \mu \, dt \\ &= \int_y^{l_1} (\lambda_1 - A^*(1)) u \mu \, dt \geq Cu(x_0) \int_y^{l_1} \mu \, dt \end{aligned} \quad (2.12)$$

where C is a constant independent of y . Now consider the behavior of r as $x \rightarrow l_1$. Two possibilities exist: either r does not vanish near l_1 or else l_1 is a limit point of zeroes of r . In the first case we may assume, moving x_0 to the right if necessary, that $r(y) \neq 0$ for $x_0 \leq y < l_1$. Since $pu'(y) > 0$, (2.12) gives

$$|ru(y)| \geq C_1 \int_y^{l_1} \mu \, dt \quad x_0 \leq y$$

with some other constant C_1 . Dividing by r , squaring and integrating we get

$$C_1^2 \int_{x_0}^{l_1} \left| r^{-1}(y) \int_y^{l_1} \mu \, dt \right|^2 \mu(y) \, dy \leq \int_{x_0}^{l_1} |u(y)|^2 \mu(y) \, dy \leq \|u\|^2 < \infty$$

which contradicts (2.8). On the other hand if l_1 is an accumulation point of zeroes of r , there exists $x_1 > x_0$ with $r(x_1) = 0$ and $p(x_1)u'(x_1) > 0$; but by (2.12)

$$-pu'(x_1) \geq C_1 \int_{x_1}^{l_1} \mu \, dt > 0$$

which is again a contradiction. The theorem is proved.

The following is an extension to the nonselfadjoint case of a classical result due to H. Weyl [6] (see also [1] and [4]).

COROLLARY 2.7. *Suppose that the measure of J is infinite at both ends. Then $D_0 = D_1$ provided that $A(1)$ and $A^*(1)$ are bounded above.*

Proof. No end point is finite. Thus (2.7) and (2.8) are (vacuously) fulfilled and Theorem 2.4 applies.

COROLLARY 2.8. *Suppose that J has finite (μ^-) -measure and that A is formally selfadjoint, with $A(1)$ bounded above and in L_μ^2 . Then $D_0 = D_1$ if and only if $1 \in D_0$.*

Proof. This is a particular case of Theorem 2.4.

3. CHANGE OF MEASURE AND APPLICATIONS

In this section we will indicate how to generalize our previous results by introducing a new measure on J . Let ν be its density. Then the map $U: f \rightarrow (\mu/\nu)^{1/2} f$ is an isometry from L_μ^2 to L_ν^2 which sends C_0^∞ onto itself and induces an operator \tilde{A} on J such that the diagram

$$\begin{array}{ccc} L_\mu^2 & \xrightarrow{(\mu/\nu)^{1/2}} & L_\nu^2 \\ A \downarrow & & \downarrow \tilde{A} \\ L_\mu^2 & \xrightarrow{(\mu/\nu)^{1/2}} & L_\nu^2 \end{array} \quad (3.1)$$

“commutes” at the C_0^∞ level. It is clear that $(\tilde{A})^* = \tilde{A}^*$, the star operations being taken with respect to ν and μ respectively. Since

$$\tilde{A}(1) = (\mu/\nu)^{1/2} A(\mu/\nu)^{-1/2} \quad (3.2)$$

$$\tilde{A}^*(1) = (\mu/\nu)^{1/2} A^*(\mu/\nu)^{-1/2} \quad (3.3)$$

we obtain from Corollary 2.7:

PROPOSITION 3.1. *Let J have infinite ν -measure at both ends. Then $D_0(A) = D_1(A)$ provided that (3.2) and (3.3) are bounded above.*

EXAMPLE 3.2 (Friedrichs, [2]). $J = (l_0, l_1)$, $-\infty < l_0 < l_1 < \infty$; $\mu = 1$. $A = (d/dx) p(d/dx) + q$ satisfies $D_0 = D_1$ if $\int^x p^{-1} dt$ is not in L^2 at either

endpoint. Set $\nu(x) = (\int^x p^{-1} dt)^2$. Then $\tilde{A}(1) = \tilde{A}^*(1) = q$. The hypothesis means that the ν -measure of J is infinite at both ends.

More generally if μ is arbitrary then $D_0 = D_1$ if q is bounded above and if $\int^x p^{-1} dt$ is not in L_μ^2 near the end points.

EXAMPLE 3.3. $A = (d/dx) x^\alpha(d/dx) + \beta x^\gamma(d/dx) + q$ on $(0, \infty)$; $\mu = 1$. Taking $\nu = x^{-1}$ we obtain

$$\begin{aligned}\tilde{A}(1) &= q - 1/2(\alpha - 3/2) x^{\alpha-2} - (\beta/2) x^{\gamma-1} \\ \tilde{A}^*(1) &= q - 1/2(\alpha - 3/2) x^{\alpha-2} - \beta(\gamma - 1/2) x^{\gamma-1}.\end{aligned}$$

These will be bounded above if

$$q(x) \leq 1/2(\alpha - 3/2) x^{\alpha-2} + C_{\beta,\gamma} x^{\gamma-1} + \text{constant}$$

where

$$C_{\beta,\gamma} = \min(\beta/2; \beta(\gamma - 1/2)) = |\beta/2| (\gamma \operatorname{sign} \beta - |\gamma - 1|). \quad (3.4)$$

This generalizes results of Sears [5] and Kurss [3] who treat the selfadjoint case ($\beta = 0$). (In fact Kurss requires $\alpha < 3/2$. The case $3/2 \leq \alpha < 2$ is of interest since q is then allowed to be $+\infty$ at $x = 0$.)

Remark. If in Example 3.3 we set $\alpha = 2$, $\beta = \gamma = 1$, and $q = 0$ we see that (3.4) holds and therefore $D_0 = D_1$. Now $x = 0$ is a finite end point for this operator in the sense of Definition 2.3. If u_0 is the corresponding function (as defined before Proposition 2.1), one can verify that the sequence $\phi_n(x) = u_0(x) u_0(1/nx)$ is in C_0^∞ and converges to u_0 both in $D_1(A)$ and in $D_1(A^*)$. Therefore all the conditions of Theorem 2.4 hold *except* for (2.8). We conjecture that as in the selfadjoint case (cf. Corollary 2.8), this condition is superfluous for the validity of Theorem 2.4, although we do not know how to prove it.

Example 3.3 also shows that if q has polynomial growth at ∞ and is bounded above near 0, there is always a function $p > 0$ such that $D_0(A_{p,q}) = D_1(A_{p,q})$ where

$$A_{p,q} = \mu^{-1}(d/dx) p(d/dx) + q. \quad (3.5)$$

It is enough to take $p = x^\alpha$ with α sufficiently large. As we shall see the restriction on the growth of q is unnecessary. To achieve this we need a further consequence of Proposition 3.1.

PROPOSITION 3.4. Let $A = (d/dx) p(d/dx) + q$ be defined on R^1 with $\mu = 1$. Assume that

$$q(x) \leq 1/2(\operatorname{sign} x) |x|^{1/2} (p |x|^{-3/2})' + \text{constant}, \quad x \text{ large}. \quad (3.6)$$

Then $D_0 = D_1$.

Proof. We use Proposition 3.1 with $\mu = 1$, $\nu = |x|^{-1}$ for large x , $\nu(x) > 0$ and C^∞ everywhere. A computation shows that

$$\tilde{A}(1) = q - 1/2(\text{sign } x) |x| (p |x|^{-3/2})'$$

if $|x|$ is large.

THEOREM 3.5. *Suppose that J has infinite μ -measure at both ends and that q is a given C^∞ function on J . Then there exists a strictly positive function $p \in C^\infty(J)$ such that with $A_{p,q}$ as defined in (3.5) one has $D_0(A_{p,q}) = D_1(A_{p,q})$.*

Proof. We change variables by integrating the equation $dy = \mu dx$. Our assumption means that y ranges over the whole of R^1 as x ranges over J . This change of variables transfers A to the operator $\tilde{A} = (d/dy) \tilde{p} \mu (d/dy) + \tilde{q}$, with \sim on the right-hand side indicating the change of variables. It is now clear that the theorem will follow if we can prove it in R^1 with $\mu = 1$. Let $q_1 \in C^\infty(R^1)$ such that $q_1(x) \geq q(x)$ and $q_1(x) > 0$ everywhere. With a non-negative $\phi \in C_0^\infty$ such that $\phi = 1$ in a neighborhood of 0, we set

$$p = \phi + 2(1 - \phi) |x|^{3/2} \int_0^x q_1(t) (\text{sign } t) t^{-1/2} dt.$$

This is a strictly positive C^∞ function which for $|x|$ large satisfies condition (3.6) of Proposition 3.4. This proves the theorem.

Our final result goes in the opposite direction. This time there is no restriction on the measure of J .

THEOREM 3.6. *Given an arbitrary everywhere positive function $p \in C^\infty(J)$ there always exists a function $q \in C^\infty(J)$ such that with $A_{p,q}$ defined as in (3.5) one has $D_0(A_{p,q}) = D_1(A_{p,q})$.*

Proof. Choose a positive density ν such that the ν -measure of J is infinite at both ends. Consider the operator

$$\tilde{A} = \nu^{-1}(d/dx) \nu \mu^{-1} p (d/dx)$$

on J , with respect to the measure νdx . We know from Corollary 2.7 that $D_0(\tilde{A}) = D_1(\tilde{A})$. If we now transfer \tilde{A} to an operator A in L_μ^2 as in Proposition 3.1 we see that $D_0(A) = D_1(A)$. A computation shows that

$$A = \mu^{-1}(d/dx) p (d/dx) + (\nu/\mu)^{1/2} \tilde{A} (\mu/\nu)^{1/2} = A_{p,q}$$

with $q = (\nu/\mu)^{1/2} \tilde{A} (\mu/\nu)^{1/2}$. The theorem is proved.

Remark. Our proof shows that in fact there is a function $q_0 \in C^\infty(J)$ such that the conclusion of the theorem holds with *any* q which is bounded above by q_0 .

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